Models of Set Theory I – Summer 2017

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Problem 13 [6 points]

1. Show that the following are definite terms.
   (a) (x, y), $x \times y$ and $f \upharpoonright x$
   (b) $TC(x)$
   (c) V_n, for every $n \in \omega$; V_ω
2. Provide a definite formula $\varphi(x)$ which holds true exactly if x is finite.
3. For a set y let $\mathcal{P}_\omega(y)$ denote the collection of finite subsets of y, and show that $\mathcal{P}_\omega(y)$ is definite.
4. For a set y let $[y]^{<\omega}$ denote the collection of finite sequences of elements of y, and show that $[y]^{<\omega}$ is definite.

Problem 14 [4 points] Show that the following terms are not definite.

1. ω_1
2. $\mathcal{P}(\omega)$

Problem 15 (Undefinability of Truth) [4 points] Show that there is no $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$ with

$$\text{ZFC} \vdash \forall c \in \text{Fml}\, \forall x\, \forall y\ [(\varphi(c,x) \wedge \varphi(c,y)) \ \rightarrow\ x = y]$$

and

$$\text{ZFC} \vdash \exists x\, \forall y\ [\psi(y) \iff x = y] \rightarrow \forall y\, [\varphi(\ulcorner \psi \urcorner, y) \iff \psi(y)]$$

for every $\in$-formula $\psi(v)$.

Hint: Show that there is an ordinal α with $\neg\varphi(n, \alpha)$ for all $n \in Fml$ and consider the least ordinal with this property.

Problem 16 [6 points] Let $\langle M_i \mid i \in \omega\rangle$ be a sequence of sets and let $\langle f_{ij} \mid 0 \leq i < j < \omega\rangle$ be a system of maps such that each f_{ij} is an embedding (=an injective homomorphism) from $\langle M_i, \in\rangle$ to $\langle M_j, \in\rangle$. Assume that the f_{ij} commute, that is whenever $i < j < k < \omega$, then $f_{ik} = f_{jk} \circ f_{ij}$.

1. Show that there is a structure $\langle M, E\rangle$ such that each $\langle M_i, \in\rangle$ embeds into $\langle M, E\rangle$ by an embedding f_i with the property that $f_i = f_j \circ f_{ij}$ whenever $i < j < \omega$. Show that there is such a structure $\langle M, E\rangle$, which is unique up to isomorphism, such that whenever $\langle N, F\rangle$ is another such structure, then there is an embedding from $\langle M, E\rangle$ to $\langle N, F\rangle$.

Hint: Consider the structure consisting of pairs $\langle i, x\rangle$ where $i < \omega$ and $x \in M_i$, and identify two of its elements $\langle i, x\rangle$ and $\langle j, y\rangle$ if $i < j$ and $f_{ij}(x) = y$. We call the resulting structure the direct limit of the sequence of sets and system of maps.

2. Provide an example of a sequence of sets and a system of maps as above, such that the corresponding direct limit $\langle M, E\rangle$ as obtained in the first part of this exercise is not well-founded, that is there is a sequence $\langle x_i \mid i < \omega\rangle$ in M such that $x_{i+1} \, E \, x_i$ for every $i < \omega$.